Logic in Action

Chapter 4: The World according to Predicate Logic

http://www.logicinaction.org/

Looking for further structure

Statement	Propositional translation
John reads	
John walks	

Looking for further structure

Statement	Propositional translation
John reads	p
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Looking for further structure

Statement	Propositional translation
John reads	p
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Looking for further structure

Statement	**Propositional translation**
John reads	$\boldsymbol{p}$
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But the fact that both statements talk about "John" is lost.

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With a language including predicates . . .

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Looking for further structure

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With a language including predicates ...

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Looking for further structure

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With a language including predicates . . .

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John reads	Rj
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The language of **predicate logic** allow us

1. to talk about objects, their properties and their relations with other objects, and
2. to make use of **universal** and **existential** quantification.

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$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$$

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5. **Quantifiers**

$\forall x$ ("**for all** x") and $\exists x$ ("**there exists an** x")

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$\forall x(Ax \rightarrow Bx)$

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Example: syllogistic statements

All *A* are *B*	$\forall x(Ax \to Bx)$
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$\forall x(Ax \to Bx)$

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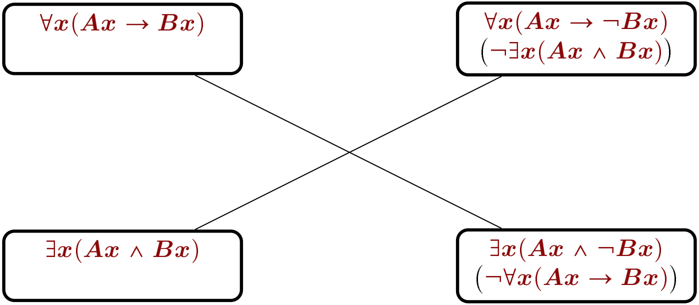

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- We can talk about **relations** between two or more objects:

John sees Mary:
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- We can express **more complicated quantifier petterns**:

Everyone sees someone:
Someone sees everyone:
Everyone is seen by someone:
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Examples

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Gx	– x is a girl
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x loves a girl $\qquad \exists y(Gy \wedge Lxy)$

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Every girl who loves all boys does not love every girl	$\forall x\Big((Gx \wedge \underline{\varphi(x)}) \to \underline{\psi(x)}\Big)$

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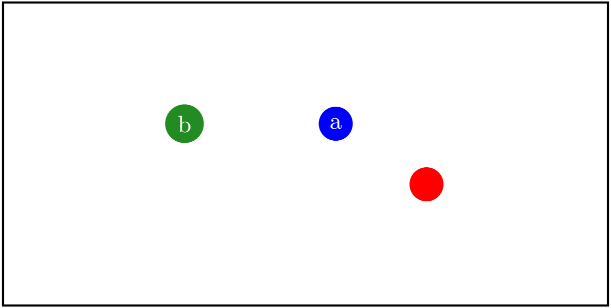
  is equivalent to $\neg\exists x\,(\varphi x \wedge \psi x)$
- $\forall x\,(\varphi x \wedge \psi x)$ is equivalent to $\forall x\,\varphi x \wedge \forall x\,\psi x$
- $\exists x\,(\varphi x \vee \psi x)$ is equivalent to $\exists x\,\varphi x \vee \exists x\,\psi x$

Evaluating predicate logic formulas (1)

Colors (***R***ed, ***G***reen, ***B***lue, ***P***urple) and shapes (***S***quare, ***C***ircle).



- Ba
- $\exists x Sx \vee Cb$
- $Ra \rightarrow Sb$
- $Ba \wedge Gb$
- $\neg Sa$
- $Ra \rightarrow \exists x Sx$

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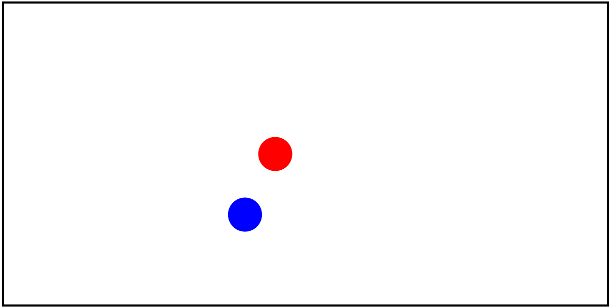

- $\exists x Rx$
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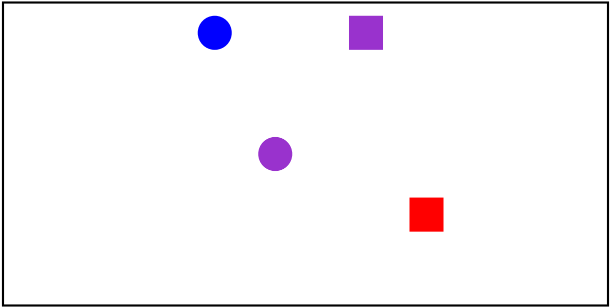
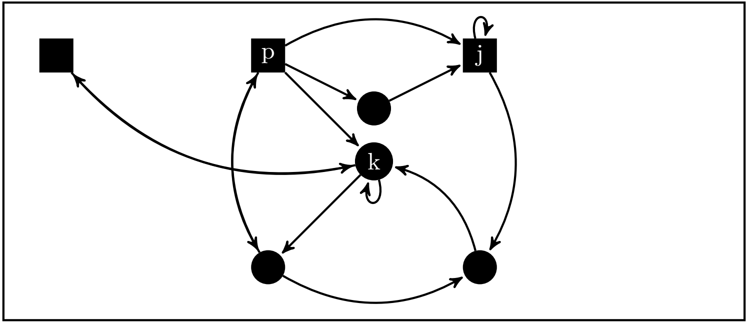


- $\exists x(Rx \wedge Cx)$
- $\forall x(Cx \vee Sx)$
- $\exists x Gx \vee \exists x Cx$
- $\exists x Rx \wedge \exists x Cx$
- $\forall x Cx \vee \forall x Sx$
- $\exists x(Gx \vee Cx)$

Evaluating predicate logic formulas (2)

■: boy ●: girl ● → ■: ● loves ■



- $Ljk \to Lkj$
- $\neg(Ljk \wedge Lkj)$
- $\forall x(Bx \to Lxk)$
- $\forall x((Bx \vee Gx) \to \neg Lxp)$
- $Ljk \wedge Lkj$
- $(Ljk \wedge Lpk) \to (\neg Lpj \wedge \neg Lkj)$
- $\neg\forall x(Gx \to Lxx)$
- $\exists x(Gx \wedge Lpx \wedge Lxj)$

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    - If φ and ψ are formulas, then the following are formulas:

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   - If φ is a formula and x is a variable, the following are formulas:

$$\forall x\varphi, \ \exists x\varphi$$

Examples of formulas

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- $\exists y \forall x(Lyx)$

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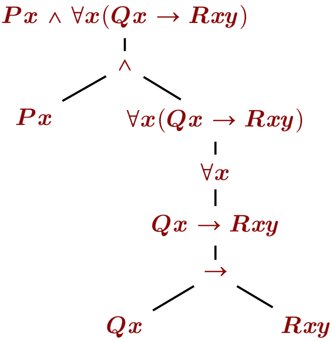
- **Bound variable.** An occurrence of a variable x is **bound** in a formula φ if there is a quantifier in φ that binds it.
- **Free variable.** An occurrence of a variable x is **free** in a formula φ if no quantifier in φ binds it.

Example

$$Px \wedge \forall x(Qx \to Rxy)$$

Example


$Px \wedge \forall x(Qx \to Rxy)$
$\wedge$
Px
$\forall x(Qx \to Rxy)$
$\forall x$
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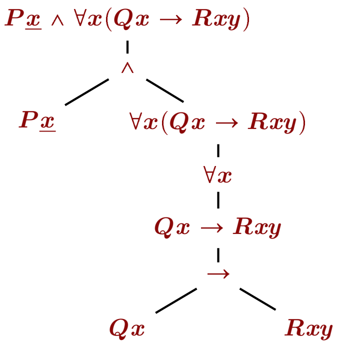
$\to$
Qx
Rxy

Example



- **This** occurrence of x is **not bound** by any quantifier (is **free**).

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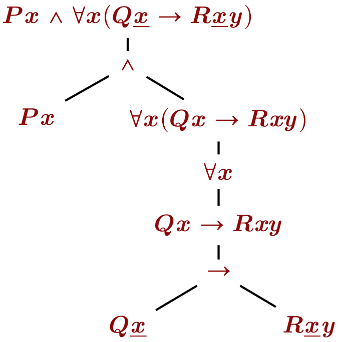

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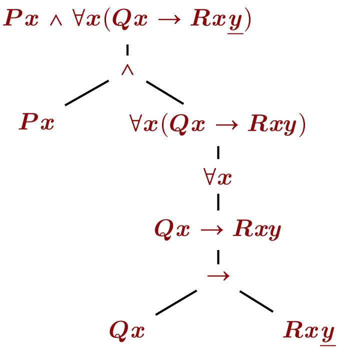

- This occurrence of x is **not bound** by any quantifier (is **free**).
- These occurrences of x are **bound** (by $\forall x$).
- **<u>This</u>** occurrence of y is **not bound** by any quantifier (is **free**).

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Models

real objects?
what is a domain?
cf. Tarski's correspondence theory

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For a predicate P, sometimes we will abbreviate relation $I(P)$ as P_I.

Modifying variable assignments

Given a **variable assignment** $\boldsymbol{g}$, a **variable** $\boldsymbol{x}$ and an **object** $\boldsymbol{d}$ in $\boldsymbol{D}$, we define the **variable assignment** $\boldsymbol{g}_{[\boldsymbol{x}:=\boldsymbol{d}]}$ as

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Given a **variable assignment** g, a **variable** x and an **object** d in D, we define the **variable assignment** $g_{[x:=d]}$ as

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The **variable assignment** $\boldsymbol{g}_{[\boldsymbol{x}:=\boldsymbol{d}]}$ differs from $\boldsymbol{g}$ only in the value of $\boldsymbol{x}$, given now by $\boldsymbol{d}$.

Assigning values to terms

Given a model $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle$, the value of every term $\boldsymbol{t}$, $[\![\boldsymbol{t}]\!]^{\boldsymbol{I}}_{\boldsymbol{g}}$, is defined as follows.

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Assigning values to terms

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Deciding truth-value of formulas

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$$\langle D, I, g\rangle \models P t_1 \cdots t_n \quad \text{iff} \quad \left([\![t_1]\!]^I_g, \ldots, [\![t_n]\!]^I_g\right) \in I(P)$$

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$\langle D, I, g\rangle \models Pt_1 \cdots t_n$ iff $([\![t_1]\!]^I_g, \ldots, [\![t_n]\!]^I_g) \in I(P)$

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$$
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$\langle D, I, g\rangle \models \forall x \varphi$	iff	for every $d \in D$ we have $\langle D, I, g_{[x:=d]}\rangle \models \varphi$

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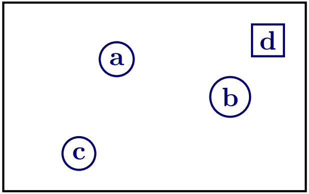
Example

Shapes (***S***quare, ***C***ircle).



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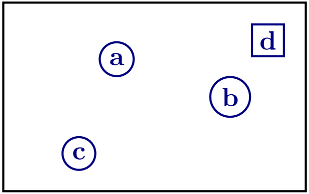

$\boldsymbol{D}$:= {ⓐ, ⓑ, ⓒ, d}

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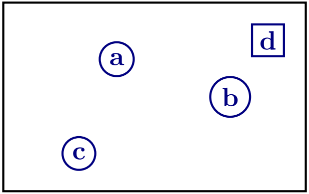

$D := \{ⓐ, ⓑ, ⓒ, \boxed{d}\}$

$I(a) := ⓐ$

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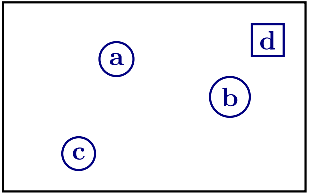

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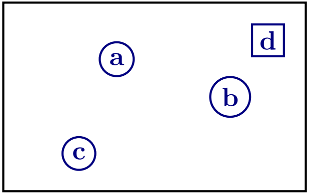

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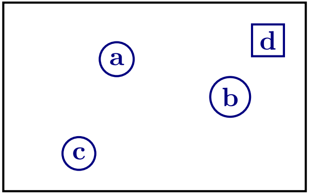

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$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \boldsymbol{Ca}$ iff

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \boldsymbol{Sx}$ iff

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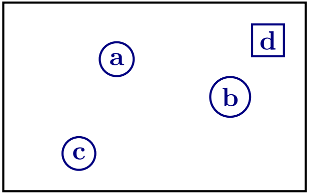

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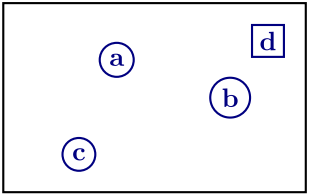

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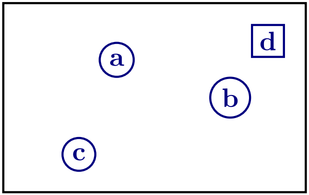

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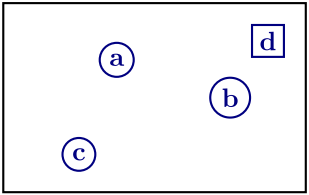

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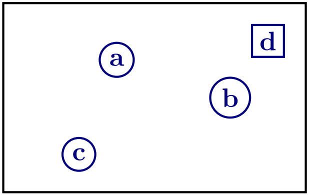

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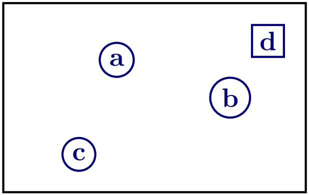

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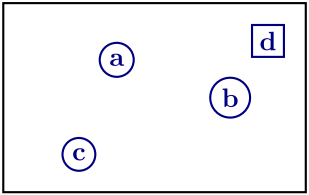

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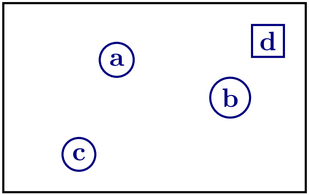
Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$\boldsymbol{I}(a) := \textcircled{a}$ $\qquad$ $\boldsymbol{I}(S) := \{\boxed{d}\}$

$\boldsymbol{I}(b) := \textcircled{b}$ $\qquad$ $\boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$

$\boldsymbol{I}(c) := \textcircled{c}$ $\qquad$ $g(x) := \textcircled{b}$

$\boldsymbol{I}(d) := \boxed{d}$ $\qquad$ $g(y) := \textcircled{a}$

$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models Sx$ iff $\textcircled{b} \in \boldsymbol{I}(S)$

$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models \exists x Sx$ iff

Example

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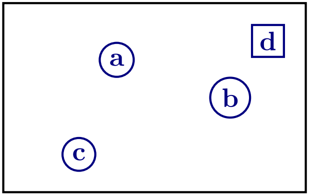

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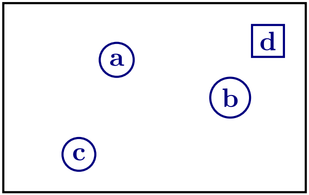

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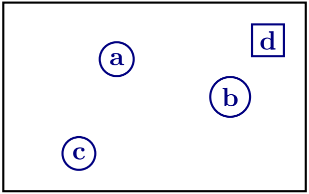

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$\langle D, I, g \rangle \models \exists x Sx$ iff there is a $o \in D$ such that $\langle D, I, g_{[x:=o]} \rangle \models Sx$

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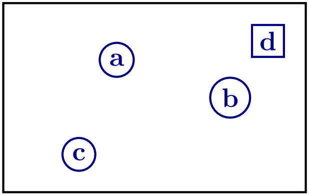

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$$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=\textcircled{a}]} \rangle \models Sx$$

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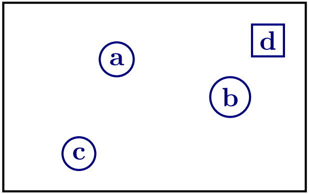

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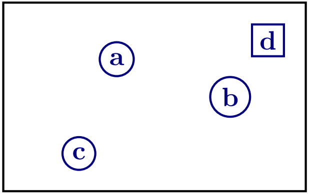
$$[\![x]\!]^{\boldsymbol{I}}_{\boldsymbol{g}_{[x:=\textcircled{a}]}} \in \boldsymbol{I}(S)$$

Example

Shapes (**S**quare, **C**ircle).



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$I(d) := \boxed{d}$ $\qquad g(y) := ⓐ$

$\langle D, I, g\rangle \models Ca$ iff $ⓐ \in \{ⓐ, ⓑ, ⓒ\}$ ✓

$\langle D, I, g\rangle \models Sx$ iff $ⓑ \in \{\boxed{d}\}$ ✗

$\langle D, I, g\rangle \models \exists x Sx$ iff there is a $o \in D$ such that $\langle D, I, g_{[x:=o]}\rangle \models Sx$

$$g_{[x:=ⓐ]}(x) \in I(S)$$

Example

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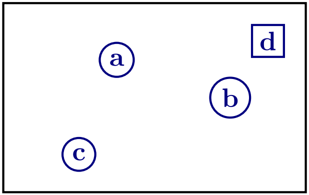

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$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \boldsymbol{Sx}$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists \boldsymbol{x}\boldsymbol{Sx}$ iff there is a $\boldsymbol{o} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[\boldsymbol{x}:=\boldsymbol{o}]} \rangle \models \boldsymbol{Sx}$

$$\textcircled{a} \in \boldsymbol{I}(\boldsymbol{S})$$

Example

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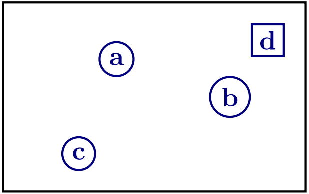

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$\langle D, I, g\rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle D, I, g\rangle \models \exists x Sx$ iff there is a $o \in D$ such that $\langle D, I, g_{[x:=o]}\rangle \models Sx$

$$\textcircled{a} \in \{\boxed{d}\}$$

Example

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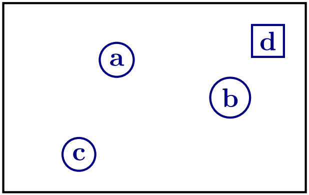

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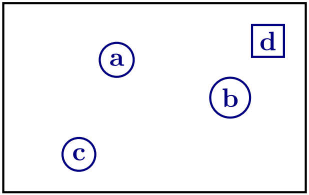

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...

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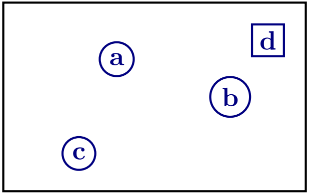

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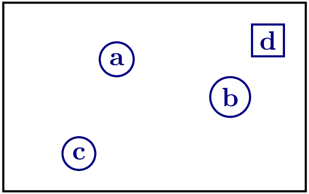

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$$[\![x]\!]^{I}_{g_{[x:=\boxed{d}]}} \in I(S)$$

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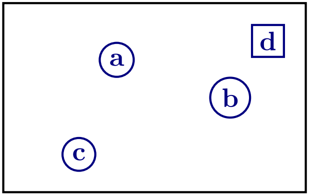

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$$\boldsymbol{g}_{[x:=\boxed{d}]}(x) \in \boldsymbol{I}(S)$$

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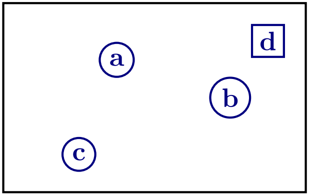

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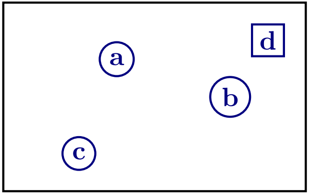

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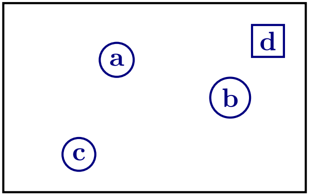

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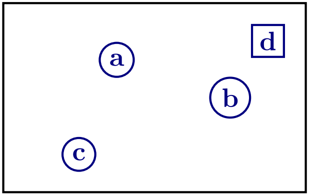

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- An **inference** $\varphi_1, \ldots, \varphi_n / \psi$ is (logically) **valid** if, for every model M for which we have $M \models \varphi_1$, ..., $M \models \varphi_n$, we also have $M \models \psi$. In such case we will write $\varphi_1, \ldots, \varphi_n \models \psi$.

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- ψ is (logically) **equivalent** to φ if

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A formula that can be derived by following these principles in a *finite* number of steps is called a **theorem**.

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Hence, $(\varphi)^x_t \to \exists x \varphi$ is a theorem.

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- $\langle D, I, g\rangle \models t_1 = t_2$ iff $[\![t_1]\!]_g^I$ and $[\![t_2]\!]_g^I$ are the same object.

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- Example: the successor function s is given by $I(s)(n) := n + 1$.